Note: Series Reversion of Stieltjes Coefficients

Benedict Irwin[1]

[1]University of Cambridge

April 28, 2020

If we take the Laurent expansion of the Riemann zeta function about $s=1$

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(s-1)^n$$

which defines γ_n, the Stieltjes constants, where γ_0 is the Euler-Mascheroni constant. Next perform a series reversion on this to give a series

$$\chi(s) = 1 + \frac{1}{s} + \sum_{n=2}^{\infty} \frac{\kappa(n)}{s^n}$$

which has expansion

$$\chi(s) = 1 + \frac{1}{s} + \frac{\gamma_0}{s^2} + \frac{\gamma_0^2-\gamma_1}{s^3} + \frac{2\gamma_0^3 - 6\gamma_0\gamma_1 + \gamma_2}{2s^4} + \cdots \tag{1}$$

The coefficients $\kappa(n)$ seem to decrease quite steadily, even up n being a few hundred, where the γ_n get large.

Letting

$$R_n = \sum_{k=1}^{n} i_k$$

and

$$P_n = \sum_{k=1}^{n} k i_k$$

and $\{i\}_n = \{i_1, i_2, \cdots | P = n-1\}$, I have observed the expression for $\kappa(n)$ from series reversion to be

$$\kappa(n) = \sum_{\{i\}} (-1)^n \left[\prod_{j=1}^{R-1} j - n\right] \left[\prod_{k=1}^{n} \frac{1}{i_k!}\left(\frac{\gamma_k}{k!}\right)^{i_{k+1}}\right] \gamma_0^{i_1}$$

where we define $\kappa(0) = 1$. Two examples

$$\kappa(3) = \gamma_0^2 - \gamma_1 = -\sum_{i_1+2i_2+3i_3=2} \left[\prod_{j=1}^{i_1+i_2+i_3-1} (j-n)\right] \frac{\left(\frac{\gamma_1}{1!}\right)^{i_2}\left(\frac{\gamma_2}{2!}\right)^{i_3}}{i_1! i_2!} \gamma_0^{i_1}$$

$$\kappa(4) = \gamma_0^3 - 3\gamma_0\gamma_1 + \frac{\gamma_2}{2} = \sum_{i_1+2i_2+3i_3+4i_4=3} \left[\prod_{j=1}^{i_1+i_2+i_3+i_4-1} (j-n)\right] \frac{\left(\frac{\gamma_1}{1!}\right)^{i_2}\left(\frac{\gamma_2}{2!}\right)^{i_3}\left(\frac{\gamma_3}{3!}\right)^{i_4}}{i_1! i_2! i_3!} \gamma_0^{i_1}$$

We can conjecture that

$$\kappa(n+1) < \kappa(n), \quad n \in \mathbb{N}^{>0}?$$

Is this perhaps a more well behaved way to look at the Stieltjes constants?

n	$\kappa(n)$
0	1.000000000
1	1.000000000
2	0.5772156649
3	0.4059937693
4	0.3135616752
5	0.2556464523
6	0.2159181431
7	0.1869526867
8	0.1648872027
9	0.1475121704
10	0.1334717457
11	0.1218874671
12	0.1121649723
13	0.1038876396
14	0.09675470803
15	0.09054358346

Table 1: The first 16 coefficients of the inverse function.