Residue of Series of Functions

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Abstract

We consider the values of residues of functions. We know the residues of some functions at their poles. For example

$$Res(\Gamma(s), -k) = \frac{(-1)^k}{k!}, k \in \mathbb{Z}$$

but then some functions are hard, for example

$$Res(e^{\Gamma(s)}, -k) = ?$$

if we consider a series expansion of $e^{\Gamma[s]}$, we get terms

$$Res\left(\sum_{l=0}^{n} \frac{\Gamma(s)^l}{l!}, -k\right)$$

for some such expansions it appears that there is a convergent limit, for large n. i.e.

$$\lim_{n\to\infty} Res\left(\sum_{l=0}^{n} \frac{\Gamma(s)^l}{l!}, s = 0\right) \approx 0.80562017...$$

and

$$\lim_{n\to\infty} Res\left(\sum_{l=0}^{n} \frac{\Gamma(s)^l}{l!}, s = -1\right) \approx -1.21469623...$$

If we call this

$$Q[e^{\Gamma(s)}](k) = \lim_{n\to\infty} Res\left(\sum_{l=0}^{n} \frac{\Gamma(s)^l}{l!}, s = -k\right)$$

then we could define a function

$$f(x) = \sum_{k=0}^{\infty} Q[e^{\Gamma(s)}](k) x^k \approx 0.806 - 1.215x + 1.02621x^2 - 0.13931294x^3 + 0.04446964782x^4 - ... \quad (1)$$

where interestingly, the higher terms are more precise with only a few terms required in the truncated version of Q.

For the function f(x), it seems like the minimum is around $x_0 = 0.658734$ with a value of $v = 0.418402$, with some likely bounds $0.401076 < v < 0.454371$ and $0.62907 < x_0 < 0.726154$.